III. Hypothesis Testing

❖ **Goal:** how to decide:

$$\left\{\begin{array}{l}\text{an event has occurred}\\ \text{a signal is present}\end{array}\right. \quad \textbf{?}$$

We need the ability to make a decision among several choices.

Basic Probability concepts
- a-priori/posteriori probability
- Bayes Rule

MAP detection

Bayes detection

Error types

Maximum likelihood criterion

Maximum error probability criterion

MinMax criterion

Neyman-Pearson criterion

Multiple hypotheses

Composite hypotheses testing

Receiver Operator Characteristic (ROC) curves

III. Hypothesis Testing

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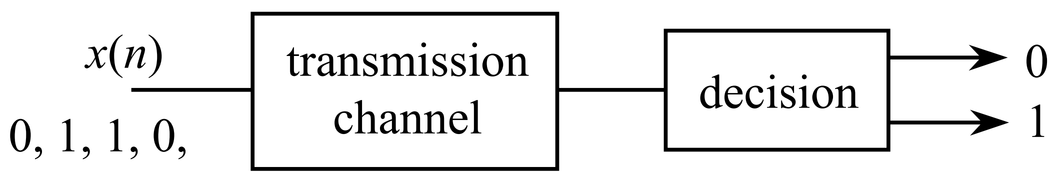
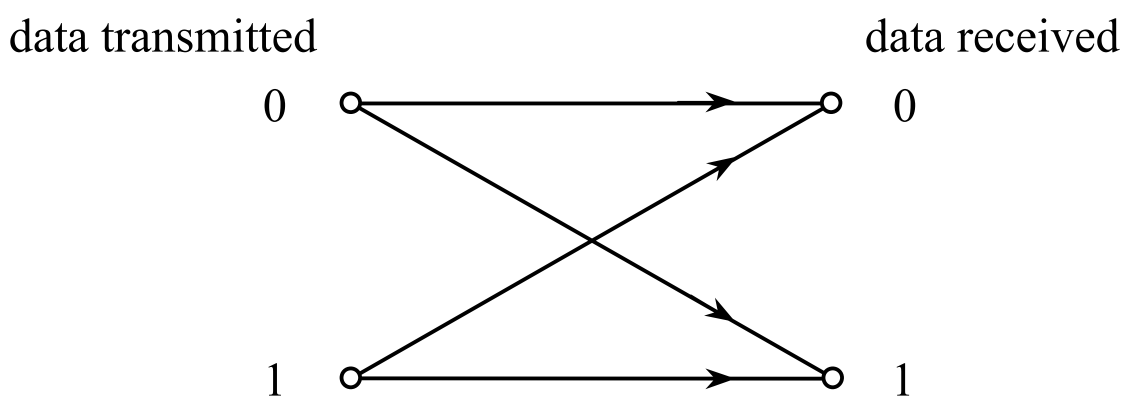


❖ **Basic Probability Concepts**



- *A priori* probability definition:

- *A posteriori* probability definition:

- **Bayes Rule for discrete events**.

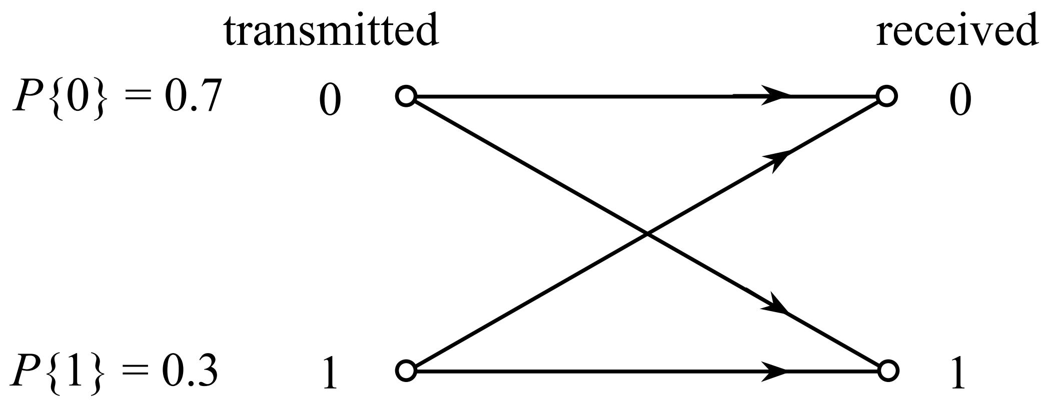
- Let $H_1, H_2, \ldots, H_N$ be a set of mutually exclusive and exhaustive events.

$$P\left(H_j \middle| A\right) =$$

$P(H_j)$:	probability of hypothesis H_j
$P(A \vert H_j)$:	conditional probability of A given hypothesis H_j
$P(H_j \vert A)$:	conditional probability that hypothesis H_j is true given event (measurement, data, received signal) A occurred

- Special case: 2 hypotheses only $\{H_i\}_{i=1,2}$

Example:



- $P\{1T|1R\} =$

- $P\{0T|0R\} =$

- $P\{0R\} =$

- $P\{1R\} =$

❖ How to formulate the problem (MAP detection)

- Assume we send a binary signal: $s = \{0,1\}$
- Assume we receive the noisy signal: $y_n = s_n + w_n$

noise

- Goal: how to detect which value of s was received.
- Problem can be formulated as distinguishing between two hypotheses.

$$\begin{cases} H_0: & y_n = w_n \\ H_1: & y_n = 1 + w_n \end{cases}$$

- Four possible outcomes:

(a)

(b)

(c)

(d)

- **How to pick a criterion for making a decision?**

  → choose hypothesis most likely to have occurred based on the observation

  → how to pick the hypothesis most probably true?

- **Decision rule:**

  – choose H_0 if:

  – choose H_1 :

- **Decision rule can be rewritten in terms of pdf:**

- **Example:** Assume you are given a transmitted bit $\{0,1\}$; received in noisy environment $\sim N(0,1/9)$

1) Compute the decision rule
2) Compute the error probabilities

❖ Bayes Detection (binary detection problem)

- Until now no particular weighting given to the two types of errors.

- Note: (1) May be cases where one error type is more harmful than the other.

  (example: radar target detection)

  (2) Cost functions may be difficult to generate.

- How to define costs:

  C_{ij}: cost associated with choosing hypothesis H_i when actually hypothesis H_j is true.

- Cost notation:

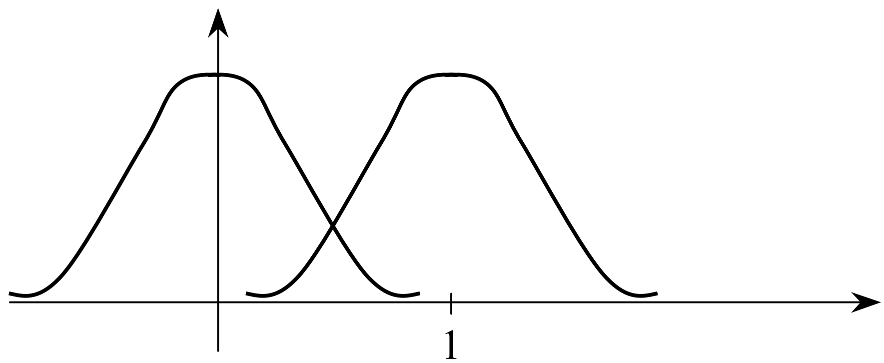
  (1) $C_{ij} \geq 0$ (positive cost implies a penalty)
  (2) usually C_{00}, C_{11} are assumed to be 0

Example: C_{00} :
C_{10} :
C_{01} :
C_{11} :

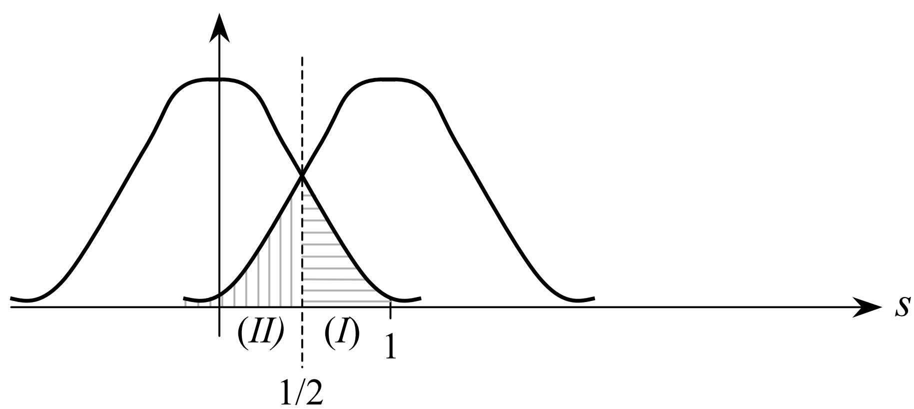
- How to compute the average cost (risk) of the decision:

$$C =$$

Back to binary signal example {0,1}



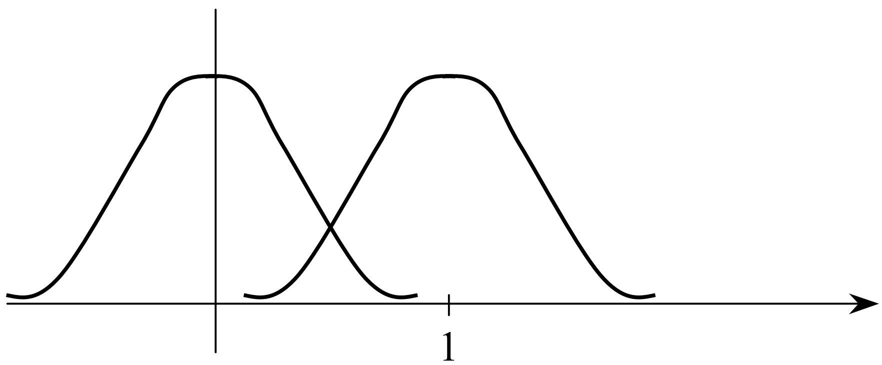
- What types of errors can be made?



Type II error	Type II error

❖ **Note**: it is not possible to reduce both errors simultaneously

→ increase one will reduce the other



Example:

Assume you transmit either "m" volts or nothing over a wire.

1) Determine an optimum decision rule to choose between the two hypotheses (based on one sample).

$$H_0: \quad y_n = w_n$$

$$n = 1, \ldots, N$$

$$H_1: \quad y_n = m + w_n$$

$$w_N \sim N(0, \sigma^2), \text{ iid}$$

2) Repeat above when using N samples to make a decision

3) Apply above results when

$$C_{00} = C_{11} = 0$$
$$C_{01} = C_{10} = K$$

$$P_0 = P_1 = 0.5$$

4) Now assume you have access to nine independent samples. Determine the optimum decision rule to choose between the two hypotheses.

Example:

- Assume you are given N samples of $y \to \{y_n\}_{n=1}^{N}$

- Assume $y_n \sim N\left(0, \sigma_0^2\right)$

  or $\qquad\qquad y_n$: i.i.d.

  $\sim N\left(0, \sigma_1^2\right)$

Determine an optimum decision rule to choose between the two hypotheses.

1) define the generic decision rule

2) apply the rule when (based on one sample only)

$$P_0 = P_1 = 0.5$$
$$\sigma_1^2 = 4, \quad \sigma_0^2 = 1$$

Example:

Assume we have an event which may or may not have occurred

event occurred: data has pdf $f_1(y) = \frac{1}{4}\exp\left(\frac{-|y|}{2}\right)$

event didn't occur: data has pdf $f_0(y) = \frac{1}{4}\exp(-|y|)$

Assume $C_{00}=C_{11}=0$; $C_{01}=C_{10}=1$; $P_0=P_1=0.5$

1) Determine the optimum decision rule

2) Determine P_D, P_{FA} , P_M

❖ Maximum Likelihood Criterion

- Assume we have no prior probability or cost information available.
- What can we do?

Example: Assume you have a constant signal of value m in AWGN $N(0,\sigma^2)$

Compute the decision rule based on one sample

❖ Maximum Error Probability Criterion

- Used in communications applications where:

$$\begin{cases} C_{01} = C_{10} = 1 \\ C_{00} = C_{11} = 0 \end{cases}$$

- Average decision cost:

Recall:

$C =$

Example:

- Assume under H_1 we observe: x=m+w and under H_0, we observe: x=w, w ~$N(0,\sigma^2)$
- Samples are observed with equal probability
- Compute the decision rule, based on a one sample basis

Example:

- Assume N independent observations of a Gaussian process are available
- under H_1 we observe: $y \sim N(m_0,1)$; and under H_0, we observe: $y \sim N(m_1,1)$, assume iid & $m_1 > m_0$
- Samples are observed with equal probability
- Compute the decision rule, based on a one sample basis

Example:

- Assume you have N independent observations of a Gaussian process.
- Assume variance is either σ_0^2 or σ_1^2 (for *message* 1 or 2).
- Design the detector which allows to distinguish between two variances.

Example:

- Assume you have N independent observations y_n of a Gaussian process.
- Assume:

$$H_0: y_n \sim N(m_0,1)$$

$$H_1: y_n \sim N(m_1,1)$$

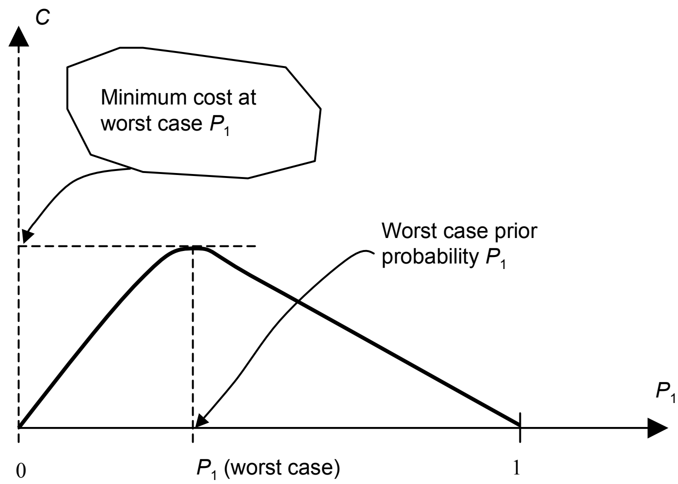
- Design the Minimum prediction error criterion detector which allows to distinguish between two data types.

❖ Min-Max (Minimax) Criterion

- Used when cost information C_{ij} is available but *a priori* probability P_0, P_1 not available.

- Average overall cost:

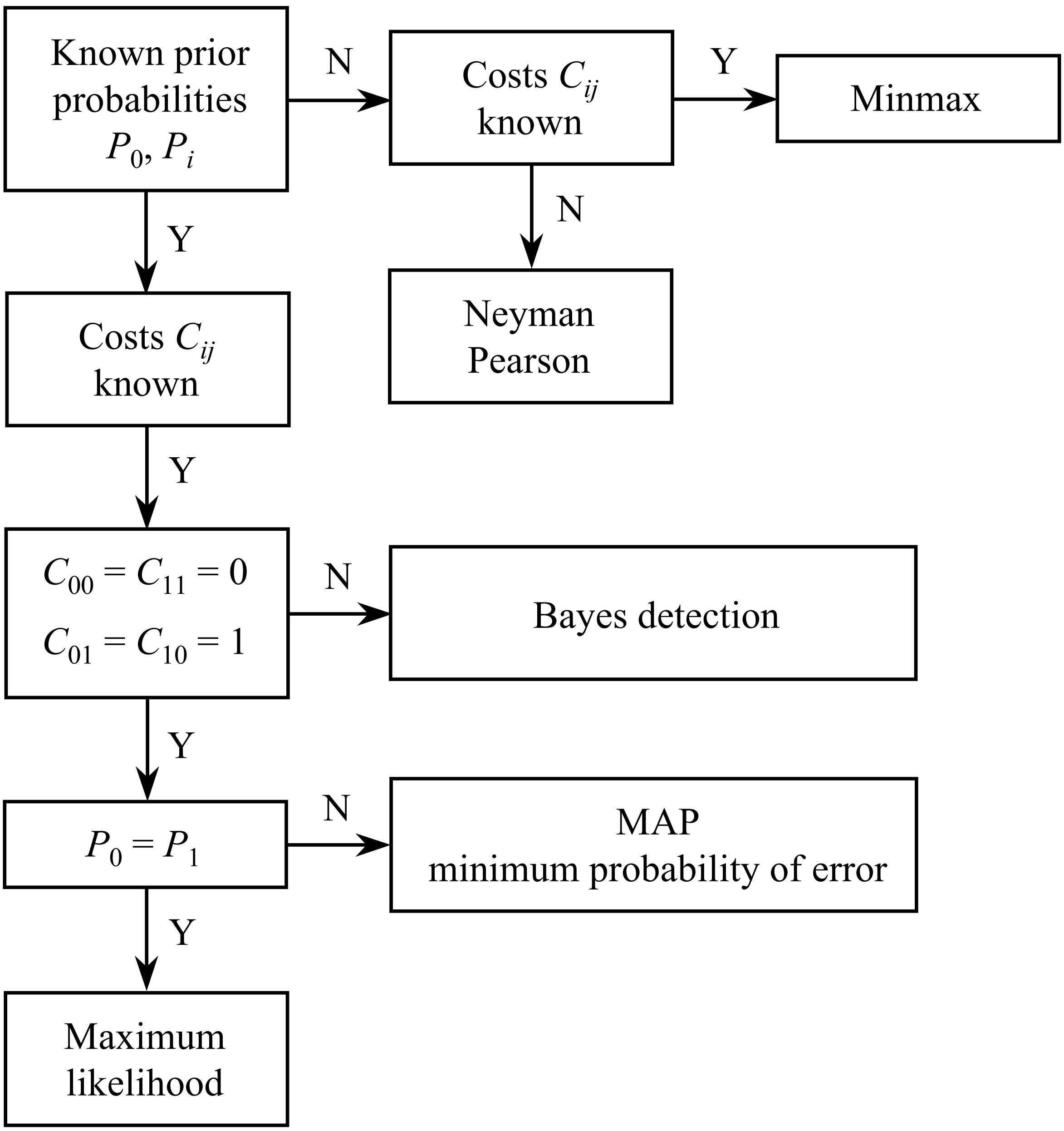
$$C =$$


C
Minimum cost at
worst case P_1
Worst case prior
probability P_1
P_1
0
P_1 (worst case)
1

Example:

- Assume N independent observations of a Gaussian process are available
- under H_1 we observe: $y = 1 + w_n$; $w_n \sim N(0,1)$; and under H_0, we observe: $y = 2 + w_n$
- Assume $C_{00} = C_{11} = 0$; $C_{01} = C_{10} = 1$;
- Compute the min-max decision rule, based on a one sample basis

Binary Hypothesis Testing Schemes



Test Name	Data Model Assumptions	Decision Rule	Optimality Criterion
Minimum probability of error (MAP)	• Hypothesis modeled as random events with known pdfs. • Known prior probabilities P_0, P_1.	$\dfrac{f_1(y)}{f_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{P_0}{P_1}$	Minimize cost $C = P_0\, P\left(D_1 \middle\| H_0\right) + P_1\, P\left(D_0 \middle\| H_0\right)$
Maximum likelihood	• Hypothesis modeled as random events with known pdfs.	$\dfrac{f_1(y)}{f_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$	
Bayes detection	• Hypothesis modeled as random events with known pdfs. • Known prior probabilities P_0, P_1. • Cost functions C_{ij} known.	$\dfrac{f_1(y)}{f_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \dfrac{P_0\left(C_{10}-C_{00}\right)}{P_1\left(C_{01}-C_{11}\right)}$	Minimize cost $C = \sum_{i,j=0}^{1} C_{ij}\, P\left(D_i \middle\| H_j\right) P\left(H_j\right)$
Minmax	• Hypothesis modeled as random events with known pdfs. • Cost functions C_{ij} known.	$\dfrac{f_1(y)}{f_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$ where γ defined so that $P_{FA} = \dfrac{C_{11}-C_{00}}{C_{10}-C_{00}} + \dfrac{C_{01}-C_{11}}{C_{10}-C_{00}} P_M$ with $P_{FA} = \int_{\gamma}^{\infty} f_0(y)\,dy$ $P_M = \int_{-\infty}^{\gamma} f_1(y)\,dy$	Minimize maximum average cost $\dfrac{\partial C}{\partial P_1} = 0 \text{ ==> }$ $\left(C_{11}-C_{00}\right) + \left(C_{01}-C_{11}\right) P_M$ $-\left(C_{10}-C_{00}\right) P_{FA} = 0$
Neyman-Pearson	• Hypothesis modeled as random events with known pdfs.	$\dfrac{f_1(y)}{f_0(y)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$ with $P_{FA} = \alpha$ user specified	Maximize probability of detection P_D for a given P_{FA}